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NOTE ON THE FUNCTIONS DEFINED BY INFINITE SERIES WHOSE TERMS ARE ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE; WITH CORRESPONDING THEOREMS FOR DEFINITE INTEGRALS.

BY W. F. OSGOOD.

LET

$$f_1(x) + f_2(x) + \cdots \qquad (a)$$

be an infinite series of real functions of the real variable x, $f_n(x)$ being single valued and continuous throughout the interval (a, b) : $a \leqq x \leqq b$, and let the series converge for all values of x pertaining to this interval. Its value shall be denoted by $F(x)$. The class of functions $F(x)$ thus defined and the mode of convergence of the series have been studied at length.* In particular, the function $F(x)$ need not be continuous throughout any interval lying in (a, b). This class is coextensive with the class which, at first sight, might appear to be smaller and which is obtained by imposing on $f_n(x)$ the further restriction that it be *analytic* throughout the interval (a, b). For, let

$$s_n(x) = f_1(x) + \cdots + f_n(x).$$

Then there exists a function $\sigma_n(x)$ analytic throughout the interval (a, b) and differing numerically from $s_n(x)$ uniformly by an arbitrarily small amount, *i. e.* $\sigma_n(x)$ satisfies the relation

$$|s_n(x) - \sigma_n(x)| < \epsilon,$$

where ϵ is an arbitrarily small positive quantity.† It is readily seen that the function $\sigma_n(x)$ converges towards the value $F(x)$; for

$$|F(x) - s_n(x)| < \epsilon', \qquad \left.\begin{matrix}\epsilon', \text{arbitrary;}\\ n, \text{suitably chosen}\end{matrix}\right\} ;$$

$$|s_n(x) - \sigma_n(x)| < \epsilon ;$$

$$\therefore \qquad |F(x) - \sigma_n(x)| < \epsilon + \epsilon'.$$

Now form the series :

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* *Cf.* Schoenflies, Bericht über die Mengenlehre, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 8, 1900, chap. 7, p. 217.

† $\sigma_n(x)$ may, in fact, be chosen as a polynomial. This follows from Weierstrass's theorem that any continuous function of a real variable can be represented by a uniformly convergent series of polynomials. *Cf.* Picard, *Traité d'analyse*, vol. 1, 1st ed., p. 258.

$$\phi_1(x) + \phi_2(x) + \cdot \cdot \cdot \cdot ,$$

where $\phi_n(x) = \sigma_n(x) - \sigma_{n-1}(x)$, $n > 1$; $\phi_1(x) = \sigma_1(x)$. Then this is the desired series; namely, a series of analytic functions having the same function $F(x)$ for its value as the original series.

If we pass from the domain of real to that of complex variables and functions, the precise analogon would be a series of complex functions, single valued and continuous throughout a two-dimensional region T of the complex z-plane and convergent at each point of this region. Such series are without interest unless their terms are analytic functions of z. We shall here consider *the class of functions*

$$F(z) = f_1(z) + f_2(z) + \cdot \cdot \cdot \cdot$$

defined by a series of functions, each of which is analytic throughout a two-dimensional region T of the complex z-plane, the series being assumed to converge for every point z lying within this region.

The principal results of the paper are stated in Theorems I and II, and are of a very general nature. The condition of uniform convergence is nowhere imposed. In fact, Theorem II presupposes only the bare convergence of the series — nothing more. The proof of Theorem II depends on a theorem in the theory of sets of points, which is quoted in No. 5, and which has already proved of value in the study of general questions in the Theory of Functions.

We observe that from Theorem II follows at once a proof of the theorem that *a function of z which is continuous throughout T cannot be represented by means of a series of functions each analytic throughout T unless it is at least in parts of T analytic.*

§1.

1. Theorem I. *If the infinite series*

$$f_1(z) + f_2(z) + \cdot \cdot \cdot \cdot , \tag{1}$$

whose terms are all functions of z, single valued and analytic throughout a region T of the complex z-plane, converges for all values of z pertaining to a set of points which is everywhere dense throughout T (and which, in particular, may be enumerable); and if, furthermore, the relation

$$| f_1(z) + \cdot \cdot \cdot + f_n(z) | \leqq G$$

holds for all points z of this set (and hence of T) and for all values of n, G

being a positive constant, then the series converges for all values of z *in* T *and the function* $F(z)$ *defined by the series:*

$$F(z) = f_1(z) + f_2(z) + \cdots$$

is analytic throughout T.

To prove the theorem, we set

$$f_n(z) = u_n(x, y) + iv_n(x, y), \qquad z = x + iy,$$

form the series of harmonic functions

$$u_1(x, y) + u_2(x, y) + \cdots, \tag{2}$$

and show that this series converges uniformly throughout the neighborhood of every interior point A of T. Hence the function $u(x, y)$ defined by (2) is harmonic in T and the series can be differentiated term by term, the resulting series converging likewise uniformly. The series of conjugate functions,

$$v_1(x, y) + v_2(x, y) + \cdots \tag{3}$$

converges for each interior point of T and the function $v(x, y)$ that it defines is conjugate to the function $u(x, y)$.

2. The details of the proof are as follows. We begin by expressing certain harmonic functions with which we have to deal by means of Poisson's Integral.* About the point A describe a circle lying wholly within T. Denote its radius by R, any interior point (x, y) by P, and any point on the circumference by Q. Further, denote the angles that AP and AQ make with the positive axis of x by ϕ and ψ respectively, and the distance AP by r. Then, if $w(x, y)$ is any function, continuous throughout the region composed of the interior and the circumference of the circle and harmonic at all points within the circle, and if the values of $w(x, y)$ along the circumference of the circle are denoted by $W(\psi)$, Poisson's Integral gives the following representation of $w(x, y)$ at the interior points of the circle:

$$w(x, y) = \frac{1}{2\pi} \int_0^{2\pi} W(\psi) \frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2} \, d\psi. \tag{4}$$

Consider the function

$$s_n(x, y) = u_1(x, y) + \cdots + u_n(x, y).$$

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* *Cf.*, for example, Picard's *Traité d'analyse*, vol. 2, p. 16.

Its partial derivatives, $\frac{\partial s_n}{\partial x}$ *and* $\frac{\partial s_n}{\partial y}$, *are finite throughout the neighborhood of the point* A. We will choose as this neighborhood, $\mathfrak{R}_A$, the interior of a circle with A as centre and radius $\frac{1}{2}R$. From (4),

$$s_n(x, y) = \frac{1}{2\pi}\int_0^{2\pi} S_n \frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2}\, d\psi;$$

hence

$$\frac{\partial s_n}{\partial x} = \frac{1}{2\pi}\int_0^{2\pi} S_n \frac{\partial}{\partial x}\frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2}\, d\psi,$$

with a similar formula for $\partial s_n/\partial y$. Differentiation under the sign of integration is here allowable.* Denote by K a quantity that is at least as large as the maximum numerical value of either of the expressions

$$\frac{\partial}{\partial x}\frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2}, \qquad \frac{\partial}{\partial y}\frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2},$$

(x, y) being any point of $\mathfrak{R}_A$ and $0 \leqq \psi < 2\pi$. Then, since $|S_n| \leqq G$ for all values of ψ, it follows that

$$\left|\frac{\partial s_n}{\partial x}\right| \leqq KG, \qquad \left|\frac{\partial s_n}{\partial y}\right| \leqq KG, \qquad q.\ e.\ d.$$

3. We are now in a position to prove that the function $s_n(x, y)$ converges uniformly toward a limit throughout the region $\mathfrak{R}_A$. The proof follows at once from a theorem which has been stated and proved for functions of a single variable by de la Vallée-Poussin.† The extension to functions of n variables is immediate. The theorem is as follows.

Let $s_n(x)$ *be a function of* x, *continuous throughout the interval* $a \leqq x \leqq b$ *and having a finite derivative‡ for all values of* x *in the interval and for* $n = 1, 2 \ldots$; *i. e.*

$$|s_n'(x)| < L$$

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* *Cf.* Picard, *Traité d'analyse*, vol. 1, p. 29.

† *Annales de la Société scientifique de Bruxelles*, première partie, vol. 17, 1893, p. 8.

‡ Instead of a finite derivative it is enough to require that the difference-quotient remain finite, *i. e.* that

$$|s_n(x + h) - s_n(x)| \leqq Nh,$$

where x and $x + h$ are any two points of the interval, $n = 1, 2, \ldots$, and N is a constant.

for all such values of x and n, L being a constant. Furthermore, let $s_n(x)$ converge toward a limit for each one of a set of values of x everywhere dense throughout this interval, when $n = \infty$. Then $s_n(x)$ converges uniformly toward a limit for the totality of values of x pertaining to the interval.

Thus the function $s_n(x, y)$ is seen to converge uniformly toward a limit, $u(x, y)$, throughout the region consisting of $\mathfrak{R}_A$ and its boundary. $s_n(x, y)$ is a harmonic function throughout $\mathfrak{R}_A$ and it takes on a continuous set of boundary values which, when $n = \infty$, converge uniformly toward a limit. The limit of $s_n(x, y)$ is, therefore, harmonic throughout $\mathfrak{R}_A$* and

$$\frac{\partial u}{\partial x} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} + \cdots\cdots.$$

This series converges uniformly. The corresponding theorems are true for the partial derivatives of $s_n(x, y)$ and $u(x, y)$ with respect to y.

4. It remains to show that the series (3) converges and defines a function conjugate to $u(x, y)$. Consider the series

$$\sum_{n=1}^{\infty} \int_{(x_0, y_0)}^{(x, y)} \frac{\partial u_n}{\partial x} dy - \frac{\partial u_n}{\partial y} dx. \tag{5}$$

Since the series

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} \frac{\partial u_n}{\partial x}, \qquad \frac{\partial u}{\partial y} = \sum_{n=1}^{\infty} \frac{\partial u_n}{\partial y}$$

converge uniformly, these series can be integrated term by term and the value of the expression (5) is

$$\bar{v}(x, y) = \int_{(x_0, y_0)}^{(x, y)} \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx,$$

which is a function conjugate to $u(x, y)$. On the other hand, the value of the general term of (5) is $v_n(x, y) - v_n(x_0, y_0)$. Hence, the series

$$\sum_{n=1}^{\infty} v_n(x, y) - v_n(x_0, y_0)$$

converges and, if (x_0, y_0) is chosen as one of the points for which the series

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* This follows from a theorem of Harnack's. *Cf.* Harnack, *Math. Annalen*, vol. 35, 1889, p. 22, or Picard, *Traité d'analyse*, vol. 2, p. 56. That such a series can be differentiated term by term and that the resulting series is uniformly convergent can readily be shown by means of the representation by Poisson's Integral.

(1) and hence the series $\sum\limits_{n=1}^{\infty} v_n(x_0, y_0)$ converges, it is seen that the series (3) converges. If $v(x, y)$ denotes its value, then

$$v(x, y) = \bar{v}(x, y) + v(x_0, y_0),$$

and the function $v(x, y)$ is conjugate to $u(x, y)$,

This completes the proof of Theorem I.

§2.

5. THEOREM II. *If the infinite series*

$$f_1(z) + f_2(z) + \cdot \cdot \cdot \cdot ,$$

whose terms are all functions of z, *single valued and analytic throughout a region* T *of the complex* z*-plane, converges for all values of* z *lying within this region, then there exists one or more analytic functions* $F_1(z)$, $F_2(z)$, . . . *such that the domain of definition of* $F_i(z)$ *coincides, in part or wholly, with a region* t_i *lying in* T *and, for all values of* z *in* t_i,

$$F_i(z) = F(z),$$

where $F(z)$ *denotes the value of the series. Every interior point of* T *either lies within some* t_i *or else a part of one or more regions* t_i *extends into the neighborhood of this point.**

The functions $F_i(z)$ *may be infinite in number. They are always enumerable.*

Let t be an arbitrary, two-dimensional region lying wholly within T. It may happen that, for a suitably chosen (large) positive quantity M, the absolute value of

$$\mathfrak{S}_n(z) = f_1(z) + \cdot \cdot \cdot \cdot f_n(z)$$

is less than M for all points z of t and for all values of n. In that case, it follows from Theorem I that the series definesa function analytic throughout t.

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* This is not the same as saying that the point lies on the boundary of one or more regions t_i. This case is included in the statement. But it may happen that the point lies on the boundary of no region t_i, but is a point of condensation of boundaries of an infinite number of regions t_i.

It may be objected that the notation $F_i(x)$, $i = 1, 2, \ldots$, assumes from the start that these functions are enumerable. No use, however, is made of this assumption in the proof, and so it is a matter of no moment. An adequate notation could be devised by allowing i to take on other than integral values, since the number of the functions $F_i(z)$ obviously does not exceed the power (Mächtigkeit) of the continuum.

In any case, I say that *it is always possible so to choose M that, throughout some two-dimensional region t' lying in t, the relation holds:*

$$|\mathfrak{S}_n(z)| \leqq M$$

for all values of z in t' and for $n = 1, 2, \ldots$, and hence $F(z)$ is analytic throughout t', $F(z)$ denoting the value of the series.

Suppose this statement were not true. Choose a set of positive quantities M_i increasing indefinitely with i. Denote the points of t at which $|\mathfrak{S}_n(z)| \leqq M_i$ for all values of n by P_i. Then the assumption is that the points of P_i nowhere in t form a two-dimensional continuum. They cannot, then, even be everywhere dense throughout any such continuum, since $|\mathfrak{S}_n(z)|$ is a continuous function of z. The points of P_i are all found in P_{i+1}. Denote by P the set of points, each of which ultimately appears in some P_i:

$$P = \lim_{i=\infty} P_i.$$

Now each point of t ultimately appears in some P_i; hence P is identical with the totality of points of t. But t is a two-dimensional continuum. We are thus led to a contradiction of a theorem of the theory of sets of points, which says:*

In a two-dimensional region t of a plane let sets of points $P_1, P_2, \ldots$ be given, which have the following properties:

1) *the points of P_i are all contained among the points of P_{i+1};*

2) *in no two-dimensional region whatever are the points of P_i everywhere dense.*

Furthermore, let

$$P = \lim_{i=\infty} P_i$$

be the totality of points that participate in the sets P_i. Then no part of P can form a two-dimensional continuum.

6. To complete the proof of the theorem, let A be any interior point of T. If A lies in a region t_i, the theorem is granted. If not, consider an arbitrary neighborhood t of A. Then there exists one or more regions t_i having points in common with t. But t is *any* neighborhood of A.

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* This theorem (with an unessential restriction) I obtained for the case of one-dimensional regions in an article on Non-Uniform Convergence and the Integration of Series Term by Term, *Am. Jour. of Math.*, vol. 19, 1897, p. 173. The proof there given is indirect. For a simple, direct proof *cf. Math. Annalen*, vol. 53, 1900, p. 462.

Finally, the regions t_i are enumerable. This is a theorem of the theory of sets that has long since been proved. In fact, the internal area* of each t_i is a positive quantity I_i, and the sum of any number of these quantities does not exceed the internal area of T itself. If, then, we choose a set of steadily decreasing quantities $\epsilon_1, \epsilon_2, \ldots$ with $\lim\limits_{n=\infty} \epsilon_n = 0$, there will be but a finite number of t_i's for each of which $I_i > \epsilon_1$. These we write in the first row of a table and arrange at pleasure in that row. Then there will be but a finite number of further t_i's for which $I_i > \epsilon_2$. These we write in the second row of the table and arrange at pleasure in that row. And so on. Each t_i must ultimately appear in some row of the table, and this proves the theorem.

7. The question arises as to whether the number of the regions t_i can ever actually be infinite. The answer is in the affirmative, as can readily be shown by means of methods due to Runge.† Thus, for example, a series can be constructed satisfying all the conditions of the theorem for the interior of the square bounded by the axes of reals and pure imaginaries and by the lines $x = 1$ and $y = 1$, whose value $F(z)$ is $\frac{1}{2}$ when $x \geqq \frac{1}{2}$; $\frac{1}{3}$, when $x \geqq \frac{1}{3}$ but $< \frac{1}{2}$; $\frac{1}{4}$, when $x \geqq \frac{1}{4}$ but $< \frac{1}{3}$; etc.

Two or more of the functions $F_i(z)$ may be capable of being deduced from one another by means of analytic continuation, so that they form parts of one and the same analytic function. This, again, can be demonstrated by Runge's methods. Thus one can, for example, construct a series convergent for the square just considered and having the value $\frac{1}{2}$ when $x \neq \frac{1}{2}$, the value 1 when $x = \frac{1}{2}$. In fact, we may draw any enumerable set of lines whatever in T, each of which has its extremities in two distinct points of the boundary of T, the lines not cutting themselves or each other or clustering about any point lying within T.‡ Then we may assume arbitrary values along these lines, provided merely that the values for a given line are such as an analytic function could take on along that line. To each of the regions t_i into which T is thus divided we may assign an arbitrary function $F_i(z)$ analytic throughout this region. Then we can construct a series whose value $F(z)$ shall correspond to the values and the functions thus assumed.

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* For the definition of the *internal area* of a set of points *cf.*, for example, Jordan, *Cours d'analyse*, vol. 1, 2d ed., 1893, §36.

† Runge, *Acta Math.*, vol. 6, 1885, p. 229.

‡ Examples in which these conditions are not fulfilled can also be readily constructed.

§3

8. By means of Theorem I, a theorem of Stieltjes may be generalized. Stieltjes proved in substance the following theorem:[*]

If

$$f_1(z) + f_2(z) + \cdots\cdot$$

is a series of functions, each single valued and analytic throughout a region T of the z-plane, and if

$$|f_1(z) + \cdots\cdot + f_n(z)| < G$$

for all values z of T and for all values of n, G being a constant; *if, furthermore, the above series converges uniformly throughout some two-dimensional region t lying in T, then the series converges throughout the entire region T and its value is a function $F(z)$ which is analytic throughout T.*

From Theorem I it follows that the requirement of *uniform convergence throughout t* can be replaced by the less restrictive one that the series *converge for each point of a set everywhere dense throughout t.* Such a set may always be so assumed as to be enumerable.

§4.

9. Theorems I and II were stated for infinite series. Corresponding theorems may be stated for definite integrals and the proofs will hold without modification.

Let $\phi(a, z)$ be a (complex) function of the real variable a in the interval $A: a_0 \leqq a \leqq a_1$ and of the complex variable z in a region T of the z-plane. For each value of a in A, $\phi(a, z)$ shall be an analytic function of z throughout T; and for each value of z in T, $\phi(a, z)$ shall be a continuous function of a throughout A.[†] Then the integral

$$\int_{a_0}^{a_1} \phi(a, z)\,da$$

will converge for each value of z in T. Since its value is invariant of the mode of division of the interval A, we may choose a special mode depending on a parameter n that passes through only positive integral values; for example, we may divide the interval into n equal parts. Let this be done and set

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[*] Stieltjes, *Annales de la Faculté des Sciences de Toulouse*, vol. 8, 1894, p. J. 56.

[†] This condition may be replaced by the broader one that $\phi(a, z)$ be an integrable function of a throughout A.

$$s_n(z) = \sum_{i=1}^{n} \phi(a_0 + i\,\Delta a,\ z)\,\Delta a, \qquad \Delta a = \frac{a_1 - a_0}{n}.$$

Then $s_n(z)$ is a function of z analytic throughout the region T for each positive integral value of n. If it satisfies the further conditions of Theorem I or II respectively, then the same conclusions will hold regarding the function

$$F(z) = \int_{a_0}^{a_1} \phi(a, z)\,da$$

as in the former case for the function $F(z)$ defined by the series.

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